Black Hole Electromagnetic Duality

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Abstract. After defining the concept of duality in the context of general n-form abelian gauge fields in $2n$ dimensions, we show by explicit example the difference between apparent but unrealizable duality transformations, namely those in $D = 4k+2$, and those, in $D = 4k$, that can be implemented by explicit dynamical generators. We then consider duality transformations in Maxwell theory in the presence of gravitation, particularly electrically and magnetically charged black hole geometries. By comparing actions in which both the dynamical variables and the charge parameters are "rotated," we show their equality for equally charged electric and magnetic black holes, thus establishing their equivalence for semiclassical processes which depend on the value of the action itself.

I begin this lecture by paying my respects to the memory of Juan José Giambiagi, to whom this conference is dedicated. Having known him since the early '60's, I have had the opportunity of understanding his importance not only through his physics (universal as many of his ideas have become) but also through the inspiration he provided in the evolution of physics research in Argentina and indeed throughout all Latin America. He was a man of great culture, with both knowledge and perspective across a wide spectrum of human ideas, and a man of great courage as I was able to observe in the dark days around 1970 when he was exiled to La Plata. He was an optimist in spite of his dark insights. We will all miss him.

An earlier important loss to Latin American physics was that of Carlos Aragone of Uruguay and Venezuela, with whom I had the pleasure of a 25 year collaboration. He was another leader of theoretical physics in our far-flung community, who twice helped create fruitful environments – in his original and in his adopted homelands.

Finally, I thank the organizers for inviting me, even though my topic is not in the mainstream of this conference. The work described here was performed in collaboration with M. Henneaux and C. Teitelboim. Indeed it builds on work first done with the latter [1] some 20 years ago! It will appear in Phys. Rev. D early in 1997 [2], and in another paper still in process, from which the general n-form discussion is drawn.

Our motivation for returning to so old a topic is its relevance to current research. I will have time here to discuss only one aspect of duality, namely its application to black hole physics, particularly that of charged black holes and their semiclassical

behavior, that is when the actions themselves (I/$\hbar$) and not just the field equations matter. Since both electrically and magnetically charge black holes can exist, investigating their equivalence in this regime is tantamount to establishing a generalized Maxwell duality in presence of sources, both electric and magnetic, as well as of a gravitational field. We will indeed show (after reviewing the flat space, sourcefree case) that the actions of magnetically and electrically (equally) charged black holes are in fact the same, a conclusion recently reached in [3] by very different means.[1]

Let me begin with some introductory notions about duality in a more general framework to show also what duality is *not*, as there are still a number of misconceptions in the literature. Consider a general (n–1)-form potential and its associated field strength $F_{1..n} \equiv \partial_n A_{1..n-1}$. [All potential and field indices are to be understood to be totally antisymmetrized and suitably normalized; also I use "mostly plus" metric signature.] The dual of a field is always defined to be

$$ {}^*F^{1..n} \equiv \frac{1}{n!}\, \epsilon^{1..n\, n+1..2n} F_{n+1..2n} \tag{1} $$

where ϵ is the Levi–Civita symbol (with $\epsilon^{01..} = +1$) in $2n$ dimensions. Clearly only in $2n$ dimensions will n-form fields be of the same rank as their duals so that one can even attempt to speak of duality transformations, let alone invariances. Now the action, field equations, and Bianchi identities for a source-free field are

$$ I = -c_n \int d^{2n}x\, F_{1..n} F^{1..n}, \qquad \partial_1 F^{1..n} = 0, \qquad \partial_1\, {}^*F^{1..n} \equiv 0 \tag{2} $$

where $c_1 = 1/2$, $c_2 = 1/4$ etc. The (source-free) field equations and Bianchi identities are of the same form so that formally any linear transformation

$$ F \to aF + b\, {}^*F \tag{3} $$

together with its dual, ${}^*F \to a{}^*F + b\, {}^{**}F$ also gives F's that obey this pair of equations. Double duality is an operation that depends on whether $n = d/2$ is even or odd, as a little reflection on the ϵ symbol verifies:

$$ {}^{**}F = F, \;\; n = 2k+1, \qquad {}^{**}F = -F, \;\; n = 2k \tag{4} $$

(this is also the reason self-duality is only realizable in the $n = 2k+1$ case). Either way, the above formal transformation is compatible with the equations. Is this symmetry shared by other physical quantities of these theories, in particular by their actions (our main interest here) and by their stress-tensors? Although it is only the Poincaré generators that are physical in flat space, the local stress tensor becomes an observable current in presence of gravity. These quantities are bilinear in the fields so they should impose more stringent conditions than the – linear –

[1] This is not directly related to the very different question of charge quantization in the *e.g.*, $\sim \hbar$ sense. Also, we will be considering here the fixed charge sectors rather than the complementary case of fixed chemical potential, but the results should carry through to that situation as well.

equations. To see most clearly what restrictions on (3) they impose let us rewrite the bilinears symmetrically in terms of F and ${}^*F$. Surprisingly, the actions and stress tensors are of the same form in all dimensions, because the scalar identity $F_{\mu ..}F^{\mu ..} \equiv -{}^*F^*_{\mu ..}F^{\mu ..}$ is by (4) dimension-independent. It then follows from (2) that

$$I = -\frac{1}{2}c_n \int d^{2n}x(F^2 - {}^*F^2) \ . \tag{5a}$$

The corresponding stress-tensors are then easily found, by varying with respect to the metric in the usual way:

$$T^\mu_\nu = \frac{1}{2}(F^{\mu ..}F_{\nu ..} + {}^*F^{\mu ..}\ {}^*F_{\nu ..}) \ . \tag{5b}$$

In accordance with conformal invariance of the action, $T^\mu_\mu = \frac{1}{2}(F^2 + {}^*F^2) \equiv 0$. In all cases, there is the same "mismatch" between the signs in the action and stresses, so that not both would seem to remain invariant under a duality transformation. The latter must be defined as either a normal rotation or a hyperbolic one rather than the general (3) to even formally keep either a sum or a difference of squares invariant. There is also no help from the fact that cross terms in the form $F{}^*F$ are total divergences and hence irrelevant to the action (apart from possible topological effects). That is, in $4k$ dimensions $F_{\mu\nu ..}\ {}^*F^{\mu\nu ..} = \partial_\mu[\epsilon^{\mu ..}A\partial A]$ is the divergence of a Chern–Simons structure, while in $4k{+}2$, $F{}^*F$ actually vanishes identically, *e.g.*, $F_\mu(\epsilon^{\mu\nu}F_\nu) \equiv 0$. So we have a paradox: the equations and identities in all dimensions are together invariant under any linear variation of F and ${}^*F$ into each other, while the action and stress tensor can seemingly never both be invariant under any transformation at all. In fact, as we will now show, none of the above considerations is even meaningful and (despite the uniformity in (5a) and (5b)) the correct answer is that Maxwell theory and its $4k$ extensions are perfectly invariant in a precise sense under duality rotations, while duality is not even definable for scalar theory and *its* ($4k{+}2$) generalizations.

The basis for those statements is the simple remark that in a dynamical theory, only transformations that can be generated by functionals of the canonical variables are even meaningful. Until the latter are given, one cannot even know what (if any) duality change is possible, let alone whether it defines an invariance. Thus, the scalar field in $D{=}2$,

$$I = -\frac{1}{2}\int d^2x\, F_\mu F^\mu \qquad F_\mu \equiv \partial_\mu\phi \qquad {}^*F^\mu \equiv \epsilon^{\mu\nu}\partial_\nu\phi \tag{6}$$

has Hamiltonian form

$$I = \int d^2x[\pi\dot{\phi} - \frac{1}{2}(\pi^2 + \phi'^2)] \ , \tag{7}$$

the field strength having components $F_0 = \dot{\phi} = \pi$, $F_1 = \phi'$. Now it is clear that there is no generator $G = \int dx\mathcal{G}(\pi,\phi)$ such that its Poisson bracket with π and ϕ' will

rotate them into each other (with either sign). For example $[G, \pi(x)] \sim \phi'(x)$ would require $G \sim \int dy\, \phi(y)\phi'(y)$ but that is clearly a total divergence and similarly for $[G, \phi'] \sim \pi$. It is easy to see (by counting signs in ϵ) that this impossibility extends to the general $D = 4k + 2$ case.[2]

Let us turn to $D = 4k$, in particular to electrodynamics in $D = 4$, our main topic. We start with a quick review of the flat space source-free sector [1]. Here the Maxwell action may be written in terms of the reduced first order conjugate variables ($\mathbf{E}$,$\mathbf{A}$) as

$$I_M[\mathbf{E}, \mathbf{A}] = \int d^4x[-\mathbf{E} \cdot \dot{\mathbf{A}} - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)] \ , \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = 0 \ , \tag{8}$$

where $\mathbf{B} \equiv \nabla \times \mathbf{A}$. In the absence of sources, the Gauss constraint says that $\mathbf{E}$ is purely transverse,

$$\mathbf{E} \equiv \boldsymbol{\nabla} \times \mathbf{Z} \tag{9}$$

and therefore only the transverse, gauge-invariant, part of $\dot{\mathbf{A}}$ survives in the kinetic term, which may be rewritten as

$$\int d^4x \; \epsilon^{ijk} \partial_j Z_k \dot{A}_i \ . \tag{10}$$

We assert, and it is easy to check, that the above reduced I_M is invariant under the rotation of the 2 dimensional vector with components $V \equiv (\mathbf{Z}, \mathbf{A})$ or its curl $W \equiv (\mathbf{E}, \mathbf{B})$ under the usual 2-dimensional rotation,

$$V' = RV \qquad \text{or} \qquad W' = RW \ , \qquad R \equiv \exp(i\sigma_2 \cos\theta) \ . \tag{11}$$

Equally important is that the generator of this transformation exists and has a very elegant "topological" (metric independent) Chern–Simons form,

$$G = -\frac{1}{2} \int d^3x \; \epsilon^{ijk} [Z_i \partial_j Z_k + A_i \partial_j A_k] \ . \tag{12}$$

The Poisson bracket or commutator of G with V or with W engenders (11) by virtue of the canonical commutation relations $[\mathbf{E}^i, \mathbf{A}'_j] = [\delta^i_j(\mathbf{r}-\mathbf{r}')]^T$ where δ^T is the usual transverse projection of the unit operator. As usual there is some asymptotic falloff to be specified; here and in curved space we take $\mathbf{A} \sim a(\Omega)r^{-1} + \mathcal{O}(r^{-2})$ and $\mathbf{E} \sim \mathbf{e}(\Omega)r^{-2} + \mathcal{O}(r^{-3})$ where $\mathbf{a}, \mathbf{e}$ depend only on solid angle.

We must now generalize the above analysis to include nontrivial geometries and charges. The former is easy: Just write the Maxwell action in the covariant first order form,[3]

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[2] Eq. (12) immediately shows that $\epsilon^{ijklm} A_{ij} \partial_k A_{lm}$ is a total divergence for even form potentials represented here by A_{ij}.

[3] This can be done in the same way for all form actions and incidentally exhibits their common Weyl invariance.

$$I_M = -\frac{1}{2}\int d^4x\left[F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2}\,F^{\mu\nu}F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}(-g)^{-1/2}\right] \tag{13}$$

where $F^{\mu\nu}$ is a contravariant tensor density to be varied independently, then insert the usual 3+1 decomposition of the metric into its spatial part g_{ij}, mixed part $g_{0i} \equiv N_i$ and time-time part $g^{00} \equiv -N^{-2}$, so that $\sqrt{-g} = N\sqrt{g}$ where $|g|$ is the 3-metric determinant. Then it immediately follows that I_M can be written as [4]

$$I_M[\mathbf{E},\mathbf{A}] = -\int d^4x[E^i\dot{A}_i + \tfrac{1}{2}\,Ng^{-1/2}g_{ij}(E^iE^j + B^iB^j) - \epsilon_{ijk}N^iE^jB^k] \tag{14}$$

where $F^{0i} \equiv E^i$ is the electric, $B^i \equiv \epsilon^{ijk}\partial_j A_k$ the magnetic, field (both are contravariant three-densities) and all metric operators are in 3-space; we have solved the Gauss constraint (still $\partial_i E^i = 0$) so that both E^i and B^i are identically transverse, $\partial_i E^i = 0 = \partial_i B^i$. Note that although it is on an arbitrary curved background space, (14) is easily seen to be invariant under (11) via the same (metric independent!) generator G of (12) since the canonical variables and kinetic term are unchanged while $(\mathbf{E}^2 + \mathbf{B}^2)$ and $\mathbf{E}\times\mathbf{B}$ are clearly locally invariant under (11).

We now turn to the black hole case and include electric and magnetic sources. To stick to the problem of interest in [3], where only the exterior solution is considered, one can still work with the source-free Maxwell equations but one must allow for non-vanishing electric and magnetic fluxes at infinity. This is possible because the spatial sections Σ have a hole. There are thus two-surfaces that are not contractible to a point, namely, the surfaces surrounding the hole (we assume for simplicity a single black hole but the analysis can straightforwardly be extended to the multi-black hole case).

Let us first dispose of a technicality when varying in presence of electrical sources or fluxes. The variation of the action under changes of E^i,

$$\delta_E I_M = -\int d^4x\delta E^i(\dot{A}_i + Ng^{-1/2}g_{ij}E^j - \epsilon_{ijk}B^jN^k), \tag{15}$$

vanishes for arbitrary variations δE^i subject to the transversality conditions[4] $\partial_i\delta E^i = 0$ and $\delta\oint_{S^2_\infty} E^i dS_i = 0$ if and only if the coefficient of δE^i in (15) fulfills the condition

$$\dot{A}_i + Ng^{-1/2}g_{ij}E^j - \epsilon_{ijk}B^jN^k = \partial_i V \tag{16}$$

where V $(\equiv A_0)$ is an arbitrary function which behaves asymptotically as $C + O(r^{-1})$: In that case, $\delta I_M = -\int d^4x\delta E^i\partial_i V = -\oint_{S^2_\infty}\delta E^i V dS_i = -C\delta(\text{electric flux})$ $= 0$. No special conditions are required, on the other hand, when varying A_i. Thus, (14) is appropriate as it stands, *i.e.*, without "improving" it by adding surface terms to the variational principle in which the competing histories all have the same given

[4] The condition $\delta\oint_{S^2_\infty} E^i dS_i = 0$ is actually a consequence of $\partial_i\delta E^i = 0$ (and of smoothness) on spatial sections with R^3-topology. We write it separately, however, because this is no longer the case if Σ has holes, as below.

electric flux at infinity and thus also the same given electric charge (here equal to zero). As pointed out in [3], it is necessary to allow the temporal component V of the vector potential to approach a non-vanishing constant at infinity since this is what happens in the black hole case if V is required to be regular on the horizon. However, as we have just shown, in order to achieve this while working with this action, it is unnecessary to keep all three components E^i of the electric field fixed at spatial infinity; only the electric flux $\oint_{S^2_\infty} E^i dS_i$ must be kept constant in the variational principle.

In the presence of a non-vanishing magnetic flux, the magnetic field is given by the expression

$$B^i = \epsilon^{ijk}\partial_j A_k + B^i_S \tag{17}$$

where B^i_S is a fixed field that carries the magnetic flux,

$$\oint_{S^2_\infty} B^i_S dS_i = 4\pi\mu, \tag{18}$$

and where $B^i_T = \epsilon^{ijk}\partial_j A_k$ is the transverse part of B^i,

$$\partial_i B^i_T = 0, \ \oint_{S^2_\infty} B^i_T dS_i = 0. \tag{19}$$

Following Dirac, we can take B^i_S to be entirely localized on a string running from the source-hole to infinity, say along the positive z-axis $\theta = 0$. We shall not need the explicit form of B^i_S in the sequel, but only to remember that for a given magnetic charge μ, B^i_S is completely fixed and hence is not a field to be varied in the action. The only dynamical components of the magnetic field B^i are still the transverse ones, *i.e.*, A_i.

One can also decompose the electric field as

$$E^i = E^i_T + E^i_L \tag{20}$$

where the longitudinal part carries all the electric flux

$$\oint_{S^2_\infty} E^i_L dS_i = 4\pi e \tag{21}$$

and the transverse field obeys

$$\partial_i E^i_T = 0, \ \oint_{S^2_\infty} E^i_T dS_i = 0 \tag{22}$$

and can thus again be written as $E^i_T = \epsilon^{ijk}\partial_j Z_k$ for some Z_k. Given the electric charge e, the longitudinal electric field is completely determined if we impose in addition, say, that it be spherically symmetric. As we have done above, we shall work with a variational principle in which we have solved Gauss's law and in which

the competing histories have a fixed electric flux $\oint_{S^2_\infty} E^i dS_i$ at infinity. This means that the longitudinal electric field is completely frozen and that only the tranverse components E^i_T or Z^i are dynamical, as for the magnetic field.

In order to discuss duality, it is convenient to treat the non-dynamical components of E^i and B^i symmetrically. To that end, one may either redefine B^i_S by adding to it an appropriate transverse part so that it shares the spherical symmetry of E^i_L, or one may redefine E^i_L by adding to it an appropriate transverse part so that it is entirely localized on the string. Both choices (or, actually, any other intermediate choice) are acceptable here. For concreteness we may take the first choice; the fields then have no string-singularity.

In the Maxwell action, E^i and B^i are now the *total* electric and magnetic fields. Since E^i_L may be taken to be time-independent (the electric charge is constant), one may replace E^i by E^i_T in the kinetic term of (14), yielding as alternative action

$$I^{e,\mu}_M[\mathbf{E}_T, \mathbf{A}] = -\int d^4x [E^i_T \dot{A}_i + \tfrac{1}{2} N g^{-1/2} g_{ij}(E^i E^j + B^i B^j) - \epsilon_{ijk} N^i E^j B^k]. \tag{23}$$

This amounts to dropping a total time derivative – equal to zero for periodic boundary conditions – and shows explicitly that the kinetic term is purely transverse. Note that there is actually a *different* action (23), hence a distinct variational principle, for each choice of e and μ, as the notation indicates.

Consider now a duality rotation acting on the transverse, dynamical variables A_i (or B^i_T) and E^i_T. Just as in the sourceless case, the kinetic term of is invariant under this transformation: it is the same kinetic term and the transformation law is the same; the surface term at the horizon in the variation vanishes because $\dot{A}_i = 0$ and $\dot{Z}_i = 0$ there.[5] Thus, if we also rotate the (non-dynamical) components of the electric and magnetic fields in the same way, that is, if we relabel the external parameters e, μ by the same 2D rotation, so that the 2-vector $Q \equiv (e, \mu)$, becomes

$$Q' = RQ \tag{24}$$

then the actions $I^{e,\mu}_M$ and $I^{e',\mu'}_M$ are equal since $\mathbf{E}$ and $\mathbf{B}$ enter totally symmetrically in the energy and momentum densities. More explicitly, if we write the longitudinal fields as $B^i_L = \mu V^i$, $E^i_L = eV^i$, then the relevant terms in (23) are just

$$-\int d^4x \{N g^{-1/2} g_{ij}[(eE^i_T + \mu B^i_T)V^j + \frac{1}{2}(e^2 + \mu^2)V^i V^j] - \epsilon_{ijk} N^i V^j (eB^k_T - \mu E^k_T)\}\ . \tag{25}$$

[5] To discuss the surface terms that arise in the variation of the action, one must supplement the asymptotic behavior of the fields at infinity specified earlier by conditions at the horizon. These are especially obvious in the Euclidean continuation, where time becomes an angular variable with the horizon sitting at the origin of the corresponding polar coordinate system. Regularity then requires that $V \equiv A_0$ and the time derivatives $\dot{A}_i$, $\dot{E}^i$ all vanish at the horizon. We assume these conditions to be fulfilled throughout.

For the mixed terms, it is clear that the field transformation is just compensated by the parameter rotation (24), while the VV term is invariant under the latter. To put it more formally, the extended duality invariance we have spelled out is one that links *different* systems, with different parameters:

$$I_M^{e,\mu}[\mathbf{E}_T, \mathbf{A}_T] = I_M^{e',\mu'}[\mathbf{E}'_T, \mathbf{A}'_T] \ , \tag{26}$$

where the primes denote the rotated values. As a special case, for the black holes without Maxwell excitations, we find equality of equally electrically and magnetic charge actions,

$$I_M^{e,0}[\mathbf{0}, \mathbf{0}] = I_M^{0,e}[\mathbf{0}, \mathbf{0}] \tag{27}$$

as also obtained, by explicit calculation of these actions, in [3]. This equality is thus not a special artifact, but reflects a general invariance property of the action appropriate to the variational principle considered here, in which the electric and magnetic fluxes are kept fixed.

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